Noether's theorem in classical mechanics revisited

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Abstract. A didatic approach of the Noether's theorem in classical mechanics is derived and used to obtain the laws of conservation.

PACS numbers: 01.30lb

1. Introduction

Noether's[1] theorem, presented in 1918, is one of the most beautiful theorems in physics. It relates symmetries of a theory with its laws of conservation. Many modern textbooks on quantum field theory present a pedagogical version of the theorem where its power is demonstrated. The interested reader is referred to the detailed discussion due to Hill[2]. Despite the great generality of this theorem, few authors present its version for classical mechanics. See for examle the work of Desloge and Karcch[3] using an approach inspired in the work of Lovelock and Hund[4]. Several authors demonstrate Noether's theorem starting from the invariance of the Lagrangian [5][6], but in this case it is not possible to obtain the energy conservation law in a natural way.

In this article, the theorem is proved imposing invariance of the action under infinitesimal transformation, openning the possibility to extend the Noether's theorem in classical mechanics to include the energy conservation.

In section 2, the Euler-Lagrange equation is rederived. In section 3 Noether's theorem is proved, in section 4 several applications are presented and in section 5 the Noether's theorem is extended and the energy conservation obtained.

2. The Euler-Lagrange Equations

We rederive the Euler-Lagrange equations of motion for sake of completness and to introduced notation.

Let us consider a system of particles with n degrees of freedom whose generalized coordinates and velocities are, respectively, q and $\dot{q}$, characterized by the Lagrangian $L(q, \dot{q}, t)$, where q is short hand for $q_1(t), q_2(t), \ldots, q_n(t)$, with the dot representing the total time derivative. When necessary for clarification, the explicit time dependence will be displayed. This simple system is used in order to place in evidence the main features of Noether's theorem.

The most general formulation of mechanics is through the principle of least action or Hamilton's principle[7, 8]: *the motion of the system from fixed time t_1 to t_2 is such that the action integral*

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t)\mathrm{d}t \tag{2.1}$$

is an minimum ‡ for the path $q(t)$ of motion. In other words, the variation of the action δS is zero for this path

$$\delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, t)\mathrm{d}t = 0. \tag{2.2}$$

Using the variation of the Lagrangian in this equation results

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i \right) \mathrm{d}t = 0, \tag{2.3}$$

‡ Actually, in order to obtain the equations of motion we can relax this restriction imposing only that S be an extremum.

where the Einstein's summation convention on repeated indices is used. The explicit form of the variations in the coordinates and velocities in (2.3) are

$$\delta q(t) = q'(t) - q(t), \tag{2.4}$$

$$\delta \dot{q}(t) = \frac{\mathrm{d}q'(t)}{\mathrm{d}t} - \frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(q'(t) - q(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\delta q(t), \tag{2.5}$$

and can be seen in figure 1.

Integrating the second term of (2.3) by parts, using (2.5) and the condition that the

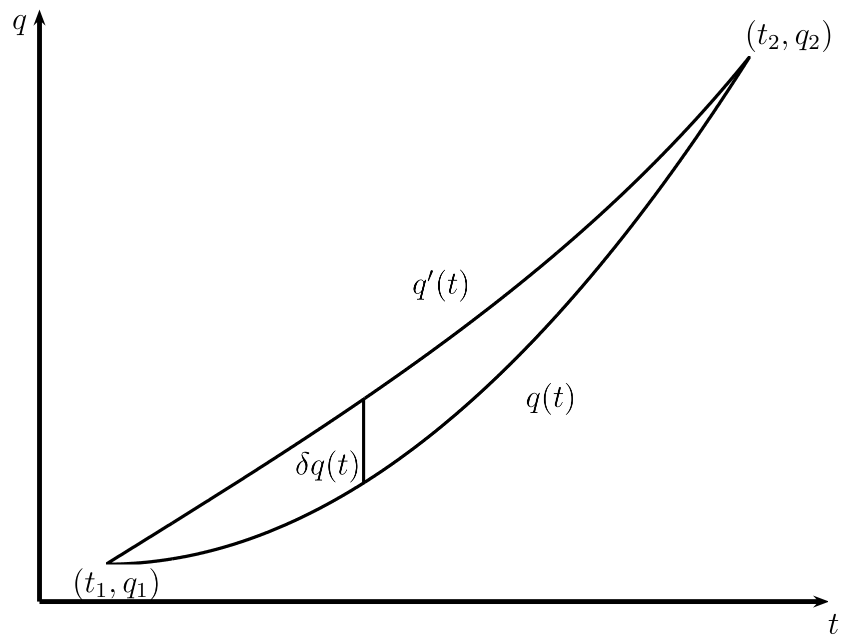

Figure 1. Varied path of the function $q(t)$

variation of the coordinates at the end points of the path t_1 and t_2 are zero:

$$\delta q(t_2) = \delta q(t_1) = 0, \tag{2.6}$$

gives

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \mathrm{d}t = 0. \tag{2.7}$$

But this is zero for an arbitrary variation δq_i only if

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \tag{2.8}$$

These are the Euler-Lagrange equations of motion.

The following should also be considered: as is well known, the action S is invariant if we replace the Lagrangian, L, of the system by a new Lagrangian, L', differing from the old one by the total time derivative of a function, $g(q, t)$, dependent only on the coordinates and the time. In fact, let us consider the new Lagrangian

$$L' = L - \frac{\mathrm{d}g(q, t)}{\mathrm{d}t}. \tag{2.9}$$

The new action is

$$S' = \int_{t_1}^{t_2} L'(q,\dot{q},t)\mathrm{d}t = \int_{t_1}^{t_2} \left(L(q,\dot{q},t) - \frac{\mathrm{d}g}{\mathrm{d}t} \right) \mathrm{d}t \tag{2.10}$$

whose variation is

$$\delta S' = \int_{t_1}^{t_2} \delta L'(q,\dot{q},t)\mathrm{d}t = \int_{t_1}^{t_2} \left(\delta L(q,\dot{q},t) - \frac{\mathrm{d}\delta g}{\mathrm{d}t} \right) \mathrm{d}t \tag{2.11}$$

but using

$$\delta g(q,t) = \frac{\partial g}{\partial q_i}\delta q_i \tag{2.12}$$

and integrating we obtain

$$\delta S' = \delta S - \frac{\partial g}{\partial q_i}(\delta q_i(t_2) - \delta q_i(t_1)). \tag{2.13}$$

Using (2.6) results in $\delta S' = \delta S$.

With the help of (2.11) we conclude that if the infinitesimal transformation that changes q to $q + \delta q$ is such that the variation of the Lagrangian can be writenn as a total time derivative of a function F:

$$\delta L = \frac{\mathrm{d}\delta g}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}t} \tag{2.14}$$

then the action S is not affected by the transformation i.e. $\delta S = 0$, and δq is a symmetry of the action.

3. Noether's Theorem

If the action of a given system is invariant under the infinitesimal transformation that changes q to $q + \delta q$, then, corresponding to this transformation there exist a law of conservation, and the conserved quantity, J, can be obtained only from de Lagrangian and the infinitesimal transformation.

In fact, let us supose that the infinitesimal transformation $q' = q + \delta q$ is a symmetry of the action, then

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i = \frac{\mathrm{d}F}{\mathrm{d}t} \tag{3.1}$$

rewritenn this equation using the Euler-Lagrange equations of motion becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{\mathrm{d}}{\mathrm{d}t}\delta q_i - \frac{\mathrm{d}F}{\mathrm{d}t} = 0 \tag{3.2}$$

but this equation can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i - F \right) = 0. \tag{3.3}$$

The expression inside the parentesis is a conserved quantity named *Noether's current*

$$J = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - F. \tag{3.4}$$

4. Several applications of the theorem

We will examine three important cases of Noether's theorem. The conservation of momentum, angular momentum and the moviment of a particle in a constant gravitational field. In the the next section we will extend the Noether's theorem to obtain the energy conservation.

4.1. Momentum conservation

Momentum conservation is obtained from the freedom we have to choose the origin of the system of coordinates. Let us consider the Lagrangian of a free point particle of mass m moving with velocity $\dot{\mathbf{x}}$,

$$L = \frac{1}{2} m \dot{x}_i \dot{x}_i. \tag{4.1}$$

Under infinitesimal space translation,

$$\left\{ \begin{array}{lllll} x'_i & = & x_i + a_i & \rightarrow & \delta x_i = a_i, \\ \dot{x}'_i & = & \dot{x}_i & \rightarrow & \delta \dot{x}_i = 0, \end{array} \right.$$

the variation of the Lagrangian becomes

$$\delta L = \frac{\partial L}{\partial x_i} \delta x_i + \frac{\partial L}{\partial \dot{x}_i} \delta \dot{x}_i = 0. \tag{4.2}$$

The first term is zero because L does not depend on x_i and the second is zero because $\delta \dot{x}_i = 0$. In this case the variation of the Lagrangian can be put in the form of (2.14) if we choose F equal a constant c. The Noether's current then results

$$J = \frac{\partial L}{\partial \dot{x}_i} \delta x_i - c = m \dot{x}_i a_i - c = const \longrightarrow p_i a_i = const. \tag{4.3}$$

As the a_i are arbitrary this is constant only if the momentum $p_i = const.$

4.2. Angular momentum conservation

Angular momentum conservation is obtained from the freedom we have to choose the orientation of the system of coordinates. Let us consider the Lagrangian of a free point particle of mass m moving with velocity $\dot{\mathbf{x}}$ in a plane

$$L = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2. \tag{4.4}$$

Under infinitesimal rotation θ,

$$\left\{ \begin{array}{lllllll} x' & = & \cos\theta x + \sin\theta y & = & x + \theta y & \rightarrow & \delta x = \theta y, \\ y' & = & -\sin\theta x + \cos\theta y & = & -\theta x + y & \rightarrow & \delta y = -\theta x \end{array} \right.$$

and

$$\left\{ \begin{array}{lllll} \dot{x}' & = & \dot{x} + \theta \dot{y} & \rightarrow & \delta \dot{x} = \theta \dot{y}, \\ \dot{y}' & = & -\theta \dot{x} + \dot{y} & \rightarrow & \delta \dot{y} = -\theta \dot{x}, \end{array} \right.$$

the variation of the Lagrangian becomes

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial \dot{y}}\delta \dot{y} = m\dot{x}\theta\dot{y} + m\dot{y}(-\theta\dot{x}) = 0. \tag{4.5}$$

Again the variation of the Lagrangian can be put in the form of (2.14) if we choose $F = c$. The Noether's current then results

$$J = \frac{\partial L}{\partial \dot{x}}\delta x + \frac{\partial L}{\partial \dot{y}}\delta y - c = const \longrightarrow (xp_y - yp_x)\theta = const. \tag{4.6}$$

As the infinitesimal angle θ is arbitrary, the expression inside the parentesis, witch is a constant, can be recognized as the component L_z of the angular momentum.

4.3. A particle in a gravitational field

Consider a particle in a constant gravitational field described by the Lagrangian

$$L = \frac{1}{2}m\dot{z}^2 - mgz. \tag{4.7}$$

Under infinitesimal space transformation

$$z' = z + a \rightarrow \delta z = a \rightarrow \delta \dot{z} = 0. \tag{4.8}$$

The variation of the Lagrangian becomes

$$\delta L = \frac{\partial L}{\partial z}\delta z + \frac{\partial L}{\partial \dot{z}}\delta \dot{z} = -mga \tag{4.9}$$

The variation of the Lagrangian can be put in the form of (2.14) if we choose

$$F = -mgat. \tag{4.10}$$

The Noether's current then results

$$J = \frac{\partial L}{\partial \dot{z}}\delta z - F = m\dot{z}a + mgat = const. \tag{4.11}$$

In the motion of a particle in a constant gravitational field the quantity $\dot{z} + gt$ wich is the initial velocity is conserved.

5. Extension of the theorem

With the formalism of the preceding section it is not possible to obtain the energy conservation. The reason comes from the fact that we have not yet defined what we mean by the variation, δt, in time, necessary to obtain the energy conservation. In order to define the variation in time let us use another parametrization for the path described by the particles. If we use a new parameter τ, the path $q = q(t)$ can be writen

$$q = q(t(\tau)) = Q(\tau), \tag{5.1}$$

$$t = T(\tau), \tag{5.2}$$

whose variations are

$$\delta q = q'(t(\tau)) - q(t(\tau)) = Q'(\tau) - Q(\tau) = \delta Q, \tag{5.3}$$

$$\delta t = T'(\tau) - T(\tau) = \delta T. \tag{5.4}$$

The action (2.1) can be writen

$$S = \int_{t_1}^{t_2} L(q,\dot{q},t)\mathrm{d}t = \int_{\tau_1}^{\tau_2} L(q,\dot{q},t)\frac{\mathrm{d}t}{\mathrm{d}\tau}\mathrm{d}\tau = \int_{\tau_1}^{\tau_2} \mathcal{L}(Q,\overset{*}{Q},T,\overset{*}{T})\mathrm{d}\tau, \tag{5.5}$$

where $\mathcal{L}$, $\overset{*}{T}$ and $\overset{*}{Q}$ are defined by the retations

$$\mathcal{L} = L\overset{*}{T}, \tag{5.6}$$

$$\overset{*}{T} = \frac{\mathrm{d}t}{\mathrm{d}\tau}, \tag{5.7}$$

$$\dot{q} = \frac{\mathrm{d}q}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{\overset{*}{Q}}{\overset{*}{T}}. \tag{5.8}$$

The Euler Lagrange equations for this action can be written

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\overset{*}{Q}} - \frac{\partial\mathcal{L}}{\partial Q} = 0, \tag{5.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\mathcal{L}}{\partial\overset{*}{T}} - \frac{\partial\mathcal{L}}{\partial T} = 0. \tag{5.10}$$

If an infinitesimal transformation leaves the action invariant then the variation of the Lagrangian can be written as a total time derivative:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial Q}\delta Q + \frac{\partial\mathcal{L}}{\partial\overset{*}{Q}}\delta\overset{*}{Q} + \frac{\partial\mathcal{L}}{\partial T}\delta T + \frac{\partial\mathcal{L}}{\partial\overset{*}{T}}\delta\overset{*}{T} = \frac{\mathrm{d}F}{\mathrm{d}\tau}. \tag{5.11}$$

Using the Euler Lagrange equations, (5.10), results in

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\overset{*}{Q}}\right)\delta Q + \frac{\partial\mathcal{L}}{\partial\overset{*}{Q}}\delta\overset{*}{Q} + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\overset{*}{T}}\right)\delta T + \frac{\partial\mathcal{L}}{\partial\overset{*}{T}}\delta\overset{*}{T} = \frac{\mathrm{d}F}{\mathrm{d}\tau} \tag{5.12}$$

or

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial\mathcal{L}}{\partial\overset{*}{Q}}\delta Q + \frac{\partial\mathcal{L}}{\partial\overset{*}{T}}\delta T - F\right) = 0. \tag{5.13}$$

Rewriting in terms of the old variables, using (5.6,5.7) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(L\overset{*}{T})}{\partial\overset{*}{Q}}\delta Q + \frac{\partial(L\overset{*}{T})}{\partial\overset{*}{T}}\delta T - F\right)\overset{*}{T} = 0. \tag{5.14}$$

With the help of (5.8) the following relations holds

$$\frac{\partial L}{\partial\overset{*}{Q}}\overset{*}{T} = \frac{\partial L}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial\overset{*}{Q}}\overset{*}{T} = \frac{\partial L}{\partial\dot{q}}, \tag{5.15}$$

$$\frac{\partial L}{\partial\overset{*}{T}}\overset{*}{T} = \frac{\partial L}{\partial\dot{q}}\frac{\partial\dot{q}}{\partial\overset{*}{T}}\overset{*}{T} = \frac{\partial L}{\partial\dot{q}}\left(-\frac{\overset{*}{Q}}{\overset{*}{T}}\right) = -\frac{\partial L}{\partial\dot{q}}\dot{q}, \tag{5.16}$$

whose substitution in (5.14), remembering that Q and T are independent variables in the new Lagrangian $\mathcal{L}$, gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L}{\partial\dot{q}}\delta q + \left(-\frac{\partial L}{\partial\dot{q}}\dot{q} + L\right)\delta t - F\right] = 0. \tag{5.17}$$

Recognizing the term inside de brackets as minus the Hamiltonian results for the Noether's current

$$J = \frac{\partial L}{\partial \dot{q}}\delta q - H\delta t - F = const. \tag{5.18}$$

5.1. Conservation of energy

Energy conservation is based on the freedom we have to choose the origin of the time. Let us cosider, as in general is the case, a Lagrangian $L(q, \dot{q}, t)$ not dependent explicitly on time. Under an infinitesimal time translation

$$\begin{cases} q' &= q \longrightarrow \delta q = 0, \\ t' &= t + \epsilon \longrightarrow \delta t = \epsilon, \end{cases}$$

the variation of the Lagrangian,

$$\delta L = \frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial L}{\partial t}\delta t = 0, \tag{5.19}$$

so, again, the variation of the Lagrangian can be put in the form of (2.14) if we choose $F = c$. The conserved current

$$J = \frac{\partial L}{\partial \dot{q}_i}\delta q_i - H\delta t - c = const \longrightarrow H\epsilon = const. \tag{5.20}$$

As ϵ is an arbitrary quantity, in order that J be constant H must be constant and can be recognized as the energy of the system. In other words, if the Lagrangian is invariant under time translation, then the energy is conserved.

6. Conclusion

The aim of this work was to present in a didatic way the Noether's theorem in the scope of Classical Mechanics. This theorem is not so important in Classical Mechanics as it is in Field Theory, this is the reason that its didatic presentation generaly comes in textbooks of Field Theory. Even in books that treat the theorem in Classical Mechanics, such as Saletan[5] and Arnold[6], do not extend the theorem to include the case of energy conservation.

We hope that this work can bring the main ideas of this theorem to undergraduate students in a clear way.

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